Oxford Cambridge and RSA

Monday 26 June 2023 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

1 A family of functions is defined as

$$f(x) = ax + \frac{x^2}{1+x}, \quad x \neq -1$$

where the parameter a is a real number. You may find it helpful to use a slider (for a) to investigate the family of curves $y = f(x)$.

(a) **(i)** On the axes in the Printed Answer Booklet, sketch the curve $y = f(x)$ in each of the following cases.

- $a = -2$
- $a = -1$
- $a = 0$ **[3]**

(ii) State a feature which is common to the curve in all three cases, $a = -2$, $a = -1$ and $a = 0$. **[1]**

(iii) State a feature of the curve for the cases $a = -2$, $a = -1$ that is **not** a feature of the curve in the case $a = 0$. **[1]**

(b) **(i)** Determine the equation of the oblique asymptote to the curve $y = f(x)$ in terms of a. **[3]**

(ii) For $b \neq -1, 0, 1$ let A be the point with coordinates $(-b, f(-b))$ and let B be the point with coordinates $(b, f(b))$.

Show that the y-coordinate of the point at which the chord to the curve $y = f(x)$ between A and B meets the y-axis is independent of a. **[3]**

(iii) With $y = f(x)$, determine the range of values of a for which

- $y \geqslant 0$ for all $x \geqslant 0$
- $y \leqslant 0$ for all $x \geqslant 0$ **[5]**

(c) In the case of $a = 0$, the curve $y = \sqrt[4]{f(x)}$ has a cusp.

Find its coordinates and fully justify that it is a cusp. **[5]**

2 Throughout this question (a, b, c) is a Pythagorean triple with the positive integers a, b, c ordered such that $a \leqslant b \leqslant c$.

(a) Show that $a^2 = b + c$ if and only if $c = b + 1$. **[4]**

(b) Create a program to find all the Pythagorean triples (a, b, c) such that $a^2 = b + c$ and $c \leqslant 1000$. Write out your program in full in the Printed Answer Booklet. **[3]**

(c) Write down the number of Pythagorean triples found by your program in **(b)**. **[1]**

(d) Prove that there is no Pythagorean triple, (a, b, c), in which $b^2 = a + c$. **[3]**

3 Wilson's theorem states that an integer $p > 1$ is prime if and only if $(p-1)! \equiv -1 \pmod{p}$.

(a) Use Wilson's theorem to show that $17! \equiv 1 \pmod{19}$. **[2]**

(b) A prime number p is called a Wilson prime if $(p-1)! \equiv -1 \pmod{p^2}$.
For example, 5 is a Wilson prime because $(5-1)! \equiv 24 \equiv -1 \pmod{25}$.
At the time of writing all known Wilson primes are less than 1000.

(i) Create a program to find all the known Wilson primes. Write out your program in full in the Printed Answer Booklet. **[4]**

(ii) Use your program to find and write down all the known Wilson primes. **[1]**

(iii) Prove that if there is an integer solution m to the equation $(p-1)! + 1 = m^2$ where p is prime, then p is a Wilson prime. **[3]**

4 In this question you are required to consider the family of differential equations

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad t \geqslant 0, \quad P(t) \geqslant 0 \ (*)$$

where r and K are positive constants. This differential equation can be used as a model for the size of a population P as a function of time t.

(a) **(i)** Determine the values of P for which

- $\frac{dP}{dt} = 0$
- $\frac{dP}{dt} > 0$
- $\frac{dP}{dt} < 0$ **[4]**

(ii) Solve the equation (*) subject to the initial condition that $P = P_0$ when $t = 0$. **[1]**

(iii) Find a property common to your solution in **(ii)** in the cases $P_0 > K$ and $P_0 < K$. **[1]**

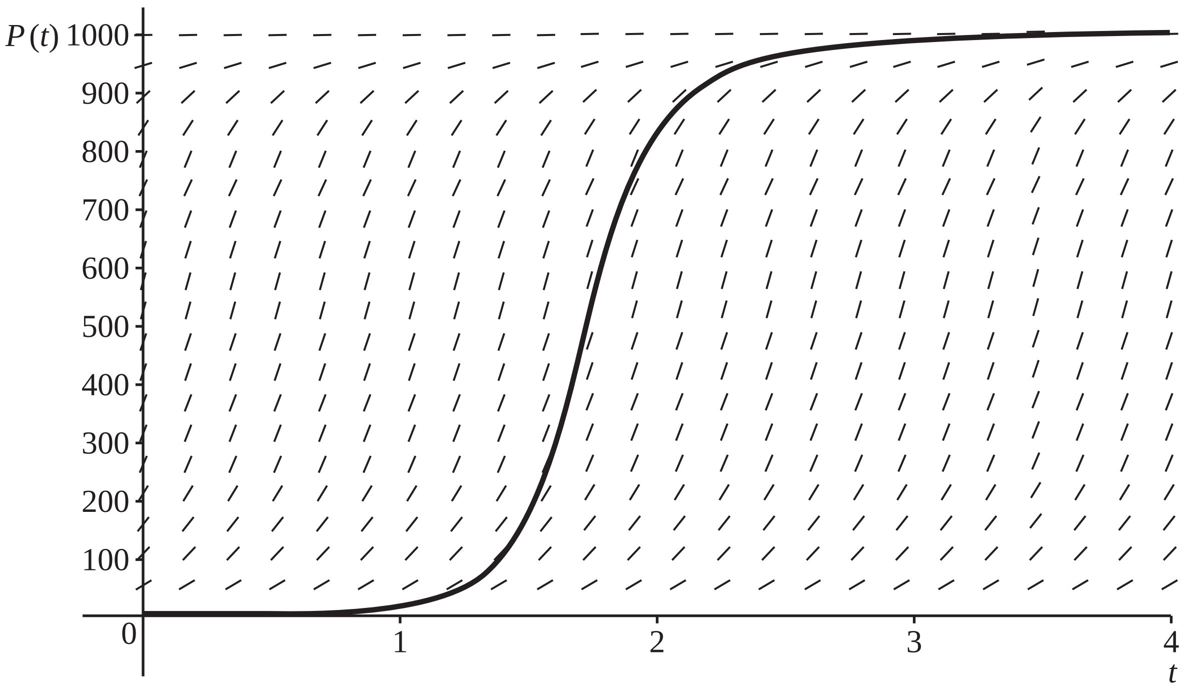
(iv) State a feature of your solution in **(iii)** for the case $P_0 > K$ which is different to the case $P_0 < K$. **[1]**

(v) Interpret the value K when $P(t)$ is the size of a population at time t. **[1]**

(b) In this question you will explore the limitations of using the Euler method to approximate solutions to the differential equation

$$\frac{dP}{dt} = 2P^{1.25}\left(1 - \frac{P}{1000}\right)^{1.5}, \ t \geqslant 0, \ P(t) \geqslant 0 \ (**)$$

The diagram shows the tangent field to (**), and a solution in which $P = 1$ when $t = 0$, produced using a much more accurate numerical method.



(i) The Euler method for the solution of the differential equation $\mathrm{f}(t, P) = \frac{\mathrm{d}P}{\mathrm{d}t}$ is as follows

$$P_{n+1} = P_n + h\mathrm{f}(t_n, P_n).$$

It is given that $t_0 = 0$ and $P_0 = 1$.

- Construct a spreadsheet to solve (**) using the Euler method so that the value of h can be varied.
- State the formulae you have used in your spreadsheet.

[4]

(ii) Use your spreadsheet with $h = 0.1$ to approximate

- $P(1)$
- $P(2)$
- $P(3)$

[1]

(iii) Use your spreadsheet with $h = 0.05$ to approximate

- $P(1)$
- $P(2)$
- $P(3)$

[1]

(iv) State, with reasons, whether the estimates to $P(t)$ given in your spreadsheet are likely to be overestimates or underestimates to the exact values. **[2]**

(v) With reference to the diagram, explain any noticeable feature identified when comparing the approximations given to $P(2)$ in **(ii)** and **(iii)**. **[2]**

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